HARDY INEQUALITIES ON METRIC MEASURE SPACES

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Dedicated to G. H. Hardy on the occasion of 100 years of his famous inequality

Abstract. In this note we give several characterisations of weights for two-weight Hardy inequalities to hold on general metric measure spaces possessing polar decompositions. Since there may be no differentiable structure on such spaces, the inequalities are given in the integral form in the spirit of Hardy's original inequality. We give examples obtaining new weighted Hardy inequalities on $\mathbb{R}^n$, on homogeneous groups, on hyperbolic spaces, and on Cartan-Hadamard manifolds.

Contents

1. Introduction 1
2. Main results 3
3. Applications and examples 5
3.1. Homogeneous groups 5
3.2. Hyperbolic spaces 6
3.3. Cartan-Hadamard manifolds 7
4. Equivalence of weight conditions 8
5. Equivalent conditions for the Hardy inequality 12
References 15

1. Introduction

In [Har20], Hardy showed his famous inequality

$$\int_b^\infty \left(\frac{\int_b^x f(t)dt}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_b^\infty f(x)^p dx, \tag{1.1}$$

where $p > 1$, $b > 0$, and $f \geq 0$ is a nonnegative function. The original discrete version of this inequality goes back to [Har18] making this year the 100^{th} anniversary of this topic, see [RS18] for a historical discussion.

Since then a lot of work has been done on Hardy inequalities in different forms and in different settings. It is clearly impossible to give a complete overview of the literature, so let us only refer to books and surveys by Opic and Kufner [OK90],

Date: October 8, 2018.

1991 *Mathematics Subject Classification.* 26D10, 22E30.

Key words and phrases. Hardy inequalities, metric measure spaces, homogeneous group, hyperbolic space, Riemannian manifolds with negative curvature.

The first author was supported in parts by the EPSRC grant EP/R003025/1 and by the Leverhulme Grant RPG-2017-151. No new data was collected or generated during the course of research.

Davies [Dav99], Kufner, Persson and Samko [KP03, KPS17], Edmunds and Evans [EE04], Mazya [Maz85, Maz11], Ghoussoub and Moradifam [GM13], Balinsky, Evans and Lewis [BEL15], and references therein.

In this paper we show that the inequality (1.1) actually holds in a much more general setting, also with rather general pairs of weights. However, the weights have to satisfy certain compatibility conditions for such inequalities to hold true, and these conditions are *necessary and sufficient*.

We note that the only known cases of our results are essentially only the Euclidean ones. The examples we give on homogeneous groups and hyperbolic spaces are new, but the main result itself is of course more general, completely characterising the weights for the integral Hardy inequality on metric measure spaces. The importance of such results is, in particular, in that they lead to *a variety of hypoelliptic Hardy-Sobolev and other inequalities, once we apply it with the weights associated to Riesz kernels (for hypoelliptic operators*, see [RY18b]).

More specifically, we consider metric spaces $\mathbb{X}$ with a Borel measure dx allowing for the following *polar decomposition* at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^\infty \int_\Sigma f(r,\omega)\lambda(r,\omega)d\omega dr, \tag{1.2}$$

for some set $\Sigma \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r,\omega) \to a$ as $r \to 0$.

The condition (1.2) is rather general since we allow the function λ to depend on the whole variable $x = (r,\omega)$. The reason to assume (1.2) is that since $\mathbb{X}$ does not have to have a differentiable structure, the function $\lambda(r,\omega)$ can not be in general obtained as the Jacobian of the polar change of coordinates. However, if such a differentiable structure exists on $\mathbb{X}$, the condition (1.2) can be obtained as the standard polar decomposition formula. In particular, let us give several examples of $\mathbb{X}$ for which the condition (1.2) is satisfied with different expressions for $\lambda(r,\omega)$:

(I) Euclidean space $\mathbb{R}^n$: $\lambda(r,\omega) = r^{n-1}$.
(II) Homogeneous groups: $\lambda(r,\omega) = r^{Q-1}$, where Q is the homogeneous dimension of the group. Such groups have been consistently developed by Folland and Stein [FS82], see also an up-to-date exposition in [FR16].
(III) Hyperbolic spaces $\mathbb{H}^n$: $\lambda(r,\omega) = (\sinh r)^{n-1}$.
(IV) Cartan-Hadamard manifolds: Let K_M be the sectional curvature on (M,g). A Riemannian manifold (M,g) is called *a Cartan-Hadamard manifold* if it is complete, simply connected and has non-positive sectional curvature, i.e., the sectional curvature $K_M \le 0$ along each plane section at each point of M. Let us fix a point $a \in M$ and denote by $\rho(x) = d(x,a)$ the geodesic distance from x to a on M. The exponential map $\exp_a : T_aM \to M$ is a diffeomorphism, see e.g. Helgason [Hel01]. Let $J(\rho,\omega)$ be the density function on M, see e.g. [GHL04]. Then we have the following polar decomposition:

$$\int_M f(x)dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))J(\rho,\omega)\rho^{n-1}d\rho d\omega,$$

so that we have (1.2) with $\lambda(\rho,\omega) = J(\rho,\omega)\rho^{n-1}$.

Throughout this paper, by $A \approx B$ we will always mean that the expressions A and B are equivalent.

The authors would like to thank Nurgissa Yessirkegenov for discussions and for checking our calculations.

2. Main results

We denote by $B(a,r)$ the ball in $\mathbb{X}$ with centre a and radius r, i.e

$$B(a,r) := \{x \in \mathbb{X} : d(x,a) < r\},$$

where d is the metric on $\mathbb{X}$. Once and for all we will fix some point $a \in \mathbb{X}$, and we will write

$$|x|_a := d(a,x).$$

Our first main result is the following characterisation of weights u and v for the corresponding Hardy inequality to hold on $\mathbb{X}$, with the characterisation for the conjugate Hardy inequality given in Theorem 2.2. The first condition is the Muckenhoupt condition while the other ones are equivalent to it.

Theorem 2.1. *Let $1 < p \leq q < \infty$ and let $s > 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition (1.2) at a. Let $u, v > 0$ be measurable functions positive a.e in $\mathbb{X}$ such that $u \in L^1(\mathbb{X}\backslash\{a\})$ and $v^{1-p'} \in L^1_{loc}(\mathbb{X})$. Denote*

$$U(x) := \int_{\mathbb{X}\backslash B(a,|x|_a)} u(y)dy$$

and

$$V(x) := \int_{B(a,|x|_a)} v^{1-p'}(y)dy.$$

Then the inequality

$$\left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)}|f(y)|dy\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left\{\int_{\mathbb{X}}|f(x)|^p v(x)dx\right\}^{\frac{1}{p}} \tag{2.1}$$

holds for all measurable functions $f : \mathbb{X} \to \mathbb{C}$ if and only if any of the following equivalent conditions hold:

(1) $\mathcal{D}_1 := \sup_{x\neq a}\left\{U^{\frac{1}{q}}(x)V^{\frac{1}{p'}}(x)\right\} < \infty.$

(2) $\mathcal{D}_2 := \sup_{x\neq a}\left\{\int_{\mathbb{X}\backslash B(a,|x|_a)} u(y)V^{q(\frac{1}{p'}-s)}(y)dy\right\}^{\frac{1}{q}} V^s(x) < \infty.$

(3) $\mathcal{D}_3 := \sup_{x\neq a}\left\{\int_{B(a,|x|_a)} u(y)V^{q(\frac{1}{p'}+s)}(y)dy\right\}^{\frac{1}{q}} V^{-s}(x) < \infty$, *provided that* $u, v^{1-p'} \in L^1(\mathbb{X})$.

(4) $\mathcal{D}_4 := \sup_{x\neq a}\left\{\int_{B(a,|x|_a)} v^{1-p'}(y)U^{p'(\frac{1}{q}-s)}(y)dy\right\}^{\frac{1}{p'}} U^s(x) < \infty.$

(5) $\mathcal{D}_5 := \sup_{x\neq a}\left\{\int_{\mathbb{X}\backslash B(a,|x|_a)} v^{1-p'}(y)U^{p'(\frac{1}{q}+s)}(y)dy\right\}^{\frac{1}{p'}} U^{-s}(x) < \infty$, *provided that* $u, v^{1-p'} \in L^1(\mathbb{X})$.

Moreover, the constant C for which (2.1) *holds and quantities $\mathcal{D}_1 - \mathcal{D}_5$ are related by*

$$\mathcal{D}_1 \leq C \leq \mathcal{D}_1(p')^{\frac{1}{p'}}p^{\frac{1}{q}}, \tag{2.2}$$

and

$$\mathcal{D}_1 \leq (\max(1,p's))^{\frac{1}{q}}\,\mathcal{D}_2,\ \mathcal{D}_2 \leq (\max(1,\frac{1}{p's}))^{\frac{1}{q}}\mathcal{D}_1,$$

$$(\frac{sp'}{1+p's})^{\frac{1}{q}}\mathcal{D}_3 \leq \mathcal{D}_1 \leq (1+sp')^{\frac{1}{q}}\mathcal{D}_3,$$

$$\mathcal{D}_1 \leq (\max(1,qs))^{\frac{1}{p'}}\mathcal{D}_4,\ \mathcal{D}_4 \leq (\max(1,\frac{1}{qs}))^{\frac{1}{p'}}\mathcal{D}_1,$$

$$(\frac{sq}{1+qs})^{\frac{1}{p'}}\mathcal{D}_5 \leq \mathcal{D}_1 \leq (1+sq)^{\frac{1}{p'}}\mathcal{D}_5.$$

In particular, Theorem 2.1 is an extension of (1.1) to the setting of metric measures spaces $\mathbb{X}$ with the polar decomposition (1.2): in particular, for $p=q$ and real-valued nonnegative measurable $f \geq 0$, inequality (2.1) becomes

$$\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^p u(x)dx \leq C\int_{\mathbb{X}} f(x)^p v(x)dx,$$

as an extension of (1.1). Indeed, in this case we can take $u(x) = \frac{1}{x^p}$, $v(x) = 1$, $\mathbb{X} = [b,\infty)$, $a = b$, so that Theorem 2.1 implies (1.1).

For the results in the case of $\mathbb{X} = \mathbb{R}$ we can refer to [KP03, PS01], and also to [PSW07] for inequalities for $q < p$. For $\mathbb{X} = \mathbb{R}^n$, the result has been proved in [Ver08], with related inequalities obtained in one dimension in [GKPW04, OK90]. For related works on hyperbolic spaces we can refer to [LY17, RY18a], and to [Ngu17, RY18a] for inequalities on Cartan-Hadamard manifolds, with the background analysis available in [GHL04, Hel01]. For the analysis of Hardy inequalities on homogeneous groups we can refer to [RS17, RSY18].

Let us also briefly discuss the conjugate Hardy inequality to that in Theorem 2.1:

Theorem 2.2. *Let $1 < p \leq q < \infty$ and $s > 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition as in* (1.2). *Let $u, v > 0$ be measurable functions positive a.e. such that $u \in L^1_{loc}(\mathbb{X})$ and $v^{1-p'} \in L^1(\mathbb{X}\backslash\{a\})$. Let*

$$U(x) = \int_{B(a,|x|_a)} u(y)dy$$

and

$$V(x) = \int_{\mathbb{X}\backslash B(a,|x|_a)} v^{1-p'}(y)dy.$$

Then the inequality

$$\left(\int_{\mathbb{X}}\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} |f(y)|dy\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left\{\int_{\mathbb{X}} |f(x)|^p v(x)dx\right\}^{\frac{1}{p}} \tag{2.3}$$

holds for all measurable functions f if and only if any of the following equivalent conditions holds:

(1) $\mathcal{D}_1^* := \sup_{x\neq a}\left\{U^{\frac{1}{q}}(x)V^{\frac{1}{p'}}(x)\right\} < \infty.$

(2) $\mathcal{D}_2^* := \sup_{x\neq a}\left\{\int_{B(a,|x|_a)} u(y)V^{q(\frac{1}{p'}-s)}(y)dy\right\}^{\frac{1}{q}} V^s(x) < \infty.$

(3) $\mathcal{D}_3^* := \sup_{x\neq a}\left\{\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)V^{q(\frac{1}{p'}+s)}(y)dy\right\}^{\frac{1}{q}} V^{-s}(x) < \infty$, *provided that* $u, v^{1-p'} \in L^1(\mathbb{X})$.

(4) $\mathcal{D}_4^* := \sup_{x\neq a}\left\{\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)U^{p'(\frac{1}{q}-s)}(y)dy\right\}^{\frac{1}{p'}} U^s(x) < \infty.$

(5) $\mathcal{D}_5^* := \sup_{x\neq a}\left\{\int_{B(a,|x|_a)} v^{1-p'}(t)U^{p'(\frac{1}{q}+s)}(t)dt\right\}^{\frac{1}{p'}} U^{-s}(x) < \infty$, *provided that* $u, v^{1-p'} \in L^1(\mathbb{X})$.

3. Applications and examples

In this section we will give examples of the application of Theorem 2.1 in the settings of homogeneous groups, hyperbolic spaces, and Cartan-Hadamard manifolds.

3.1. **Homogeneous groups.** Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q, equipped with a quasi-norm $|\cdot|$. For the general description of the setup of homogeneous groups we refer to [FS82] or [FR16]. Particular example of homogeneous groups are the Euclidean space $\mathbb{R}^n$ (in which case $Q = n$), the Heisenberg group, as well as general stratified groups (homogeneous Carnot groups) and graded groups.

In relation to the notation of this paper, let us take $a = 0$ to be the identity of the group $\mathbb{G}$. We can also simplify the notation denoting $|x|_a$ by $|x|$, which is consistent with the notation for the quasi-norm $|\cdot|$.

If we take power weights

$$u(x) = |x|^\alpha \text{ and } v(x) = |x|^\beta,$$

then the inequality (2.1) holds for $1 < p \le q < \infty$ if and only if

$$\mathcal{D}_1 = \sup_{r>0}\left(\sigma\int_r^\infty \rho^\alpha\rho^{Q-1}d\rho\right)^{\frac{1}{q}}\left(\sigma\int_0^r \rho^{\beta(1-p')}\rho^{Q-1}d\rho\right)^{\frac{1}{p'}} < \infty,$$

where σ is the area of the unit sphere in $\mathbb{G}$ with respect to the quasi-norm $|\cdot|$. For this supremum to be well-defined we need to have $\alpha + Q < 0$ and $\beta(1-p') + Q > 0$. Consequently, we have

$$\mathcal{D}_1 = \sigma^{(\frac{1}{q}+\frac{1}{p'})}\sup_{r>0}\left(\int_r^\infty \rho^{\alpha+Q-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^r \rho^{\beta(1-p')+Q-1}d\rho\right)^{\frac{1}{p'}}$$

$$= \sigma^{(\frac{1}{q}+\frac{1}{p'})}\sup_{r>0}\frac{r^{\frac{\alpha+Q}{q}}}{|\alpha+Q|^{\frac{1}{q}}}\frac{r^{\frac{\beta(1-p')+Q}{p'}}}{(\beta(1-p')+Q)^{\frac{1}{p'}}},$$

which is finite if and only if the power of r is zero. Summarising, we obtain

Corollary 3.1. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q, equipped with a quasi-norm $|\cdot|$. Let $1 < p \le q < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then the inequality*

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)}|f(y)|dy\right)^q |x|^\alpha dx\right)^{\frac{1}{q}} \le C\left\{\int_{\mathbb{G}}|f(x)|^p|x|^\beta dx\right\}^{\frac{1}{p}} \tag{3.1}$$

holds for all measurable functions $f : \mathbb{G} \to \mathbb{C}$ if and only if $\alpha+Q < 0$, $\beta(1-p')+Q > 0$ and $\frac{\alpha+Q}{q} + \frac{\beta(1-p')+Q}{p'} = 0$. Moreover, the constant C for (3.1) satisfies

$$\frac{\sigma^{\frac{1}{q}+\frac{1}{p'}}}{|\alpha+Q|^{\frac{1}{q}}(\beta(1-p')+Q)^{\frac{1}{p'}}} \le C \le (p')^{\frac{1}{p'}}p^{\frac{1}{q}}\frac{\sigma^{\frac{1}{q}+\frac{1}{p'}}}{|\alpha+Q|^{\frac{1}{q}}(\beta(1-p')+Q)^{\frac{1}{p'}}}, \tag{3.2}$$

where σ is the area of the unit sphere in $\mathbb{G}$ with respect to the quasi-norm $|\cdot|$.

3.2. **Hyperbolic spaces.** Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a \in \mathbb{H}^n$. Let us take the weights

$$u(x) = (\sinh|x|_a)^\alpha \text{ and } v(x) = (\sinh|x|_a)^\beta.$$

Then, passing to polar coordinates, $\mathcal{D}_1$ is equivalent to

$$\mathcal{D}_1 \simeq \sup_{|x|_a>0}\left(\int_{|x|_a}^\infty (\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}.$$

For the integrability of the first and the second terms we need, respectively, $\alpha+n-1 < 0$ and $\beta(1-p')+n > 0$.

Let us now analyse conditions for this supremum to be finite. For $|x|_a \gg 1$, it can be written as

$$\sup_{|x|_a\gg1}\left(\int_{|x|_a}^\infty (\exp\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}$$
$$\simeq \sup_{|x|_a\gg1}(\exp|x|_a)^{\left(\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}\right)},$$

which is finite if and only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'} \le 0$. For $|x|_a \ll 1$, it can be written as

$$\sup_{|x|_a\ll1}\left(\int_{|x|_a}^\infty (\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}\rho^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}$$
$$\simeq \sup_{|x|_a\ll1}\left(\int_{|x|_a}^R (\sinh\rho)^{\alpha+n-1}d\rho+\int_R^\infty(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}\rho^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}.$$

For some small R we have $\sinh\rho_{|x|_a\le\rho<R} \approx \rho$, so that the above supremum is

$$\approx \sup_{|x|_a\ll1}\left(|x|_a^{\alpha+n}+C_R\right)^{\frac{1}{q}}|x|_a^{\frac{\beta(1-p')+n}{p'}}.$$

Now, for $\alpha+n \ge 0$, this is

$\approx \sup_{|x|_a\ll1}|x|_a^{\frac{\beta(1-p')+n}{p'}}$, which is finite if and only if $\frac{\beta(1-p')+n}{p'} \ge 0$. At the same time, for for $\alpha+n < 0$ it is

$\approx \sup_{|x|_a \ll 1} |x|_a^{\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}}$, which is finite if and only if $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \geq 0$.
Summarising, we obtain the following

Corollary 3.2. *Let $\mathbb{H}^n$ be the hyperbolic space, $a \in \mathbb{H}^n$, and let $|x|_a$ denote the hyperbolic distance from x to a. Let $1 < p \leq q < \infty$ and let $\alpha, \beta \in \mathbb{R}$. Then the inequality*

$$\left(\int_{\mathbb{H}^n}\left(\int_{B(a,|x|_a)} |f(y)|dy\right)^q (\sinh |x|_a)^\alpha dx\right)^{\frac{1}{q}} \leq C\left\{\int_{\mathbb{H}^n} |f(x)|^p (\sinh |x|_a)^\beta dx\right\}^{\frac{1}{p}}$$

holds for all measurable functions $f : \mathbb{H}^n \to \mathbb{C}$ if and only if the parameters satisfy either of the following conditions

(A) *for $\alpha+n \geq 0$, if and only if $\alpha+n < 1$, $\beta(1-p')+n > 0$ and $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \leq \frac{1}{q}+\frac{1}{p'}$;*

(B) *for $\alpha+n < 0$, if and only if $\beta(1-p')+n > 0$ and $0 \leq \frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'} \leq \frac{1}{q}+\frac{1}{p'}$.*

3.3. **Cartan-Hadamard manifolds.** Let (M, g) be a Cartan-Hadamard manifold and assume that the sectional curvature K_M is constant. In this case it is known that $J(t, \omega)$ is a function of t only. More precisely, if $K_M = -b$ for $b \geq 0$, then $J(t, \omega) = 1$ if $b = 0$, and $J(t, \omega) = (\frac{\sinh \sqrt{b}t}{\sqrt{b}t})^{n-1}$ for $b > 0$, see e.g. [Ngu17].

When $b = 0$, then let us take $u(x) = |x|_a^\alpha$ and $v(x) = |x|_a^\beta$, then the inequality (2.1) holds for $1 < p \leq q < \infty$ if and only if

$$\sup_{|x|_a>0}\left(\int_{M\backslash B(a,|x|_a)} |y|_a^\alpha dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} |y|_a^{\beta(1-p')} dy\right)^{\frac{1}{p'}} < \infty.$$

After changing to the polar coordinates, this is equivalent to

$$\sup_{|x|_a>0}\left(\int_{|x|_a}^\infty \rho^{\alpha+n-1} d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a} \rho^{\beta(1-p')+n-1} d\rho\right)^{\frac{1}{p'}},$$

which is finite if and only if conditions of Corollary 3.1 hold with $Q = n$ (which is natural since the curvature is zero).

When $b > 0$, let us take $u(x) = (\sinh \sqrt{b}|x|_a)^\alpha$ and $v(x) = (\sinh \sqrt{b}|x|_a)^\beta$. Then he inequality (2.1) holds for $1 < p \leq q < \infty$ if and only if

$$\sup_{|x|_a>0}\left(\int_{\mathbb{M}\backslash B(a,|x|_a)} (\sinh \sqrt{b}|y|_a)^\alpha dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} (\sinh \sqrt{b}|y|_a)^{\beta(1-p')} dy\right)^{\frac{1}{p'}} < \infty.$$

After changing to the polar coordinates, this supremum is equivalent to

$$\sup_{|x|_a>0}\Big(\int_{|x|_a}^\infty (\sinh \sqrt{b}t)^\alpha (\frac{\sinh \sqrt{b}t}{\sqrt{b}t})^{n-1} t^{n-1} dt\Big)^{\frac{1}{q}}$$

$$\times\Big(\int_0^{|x|_a} (\sinh \sqrt{b}t)^{\beta(1-p')} (\frac{\sinh \sqrt{b}t}{\sqrt{b}t})^{n-1} t^{n-1} dt\Big)^{\frac{1}{p'}}$$

$$\simeq \sup_{|x|_a>0}\left(\int_{|x|_a}^\infty (\sinh \sqrt{b}t)^{\alpha+n-1} dt\right)^{\frac{1}{q}}\left(\int_0^{|x|_a} (\sinh \sqrt{b}t)^{\beta(1-p')+n-1} dt\right)^{\frac{1}{p'}},$$

which has the same conditions for finiteness as the case of the hyperbolic space in Corollary 3.2 (which is also natural since it is the negative constant curvature case).

4. Equivalence of weight conditions

In this section we prove that the quantities $\mathcal{D}_1$–$\mathcal{D}_5$ involving the weights in Theorem 2.1 are equivalent. However, it will be convenient to formulate it in the following slightly more general form:

Theorem 4.1. *Let $\alpha, \beta, s > 0$ and let $f \in L^1(\mathbb{X}\backslash\{a\})$, $g \in L^1_{loc}(\mathbb{X})$, be such that $f, g > 0$ are positive a.e in $\mathbb{X}$. Let us denote*

$$F(x) := \int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)dy,$$

and

$$G(x) := \int_{B(a,|x|_a)} g(y)dy.$$

Then the following quantities are equivalent:

(1) $\mathcal{A}_1 := \sup_{x\neq a} A_1(x;\alpha,\beta) := \sup_{x\neq a} F^\alpha(x)G^\beta(x)$.

(2) $\mathcal{A}_2 := \sup_{x\neq a} A_2(x;\alpha,\beta,s) := \sup_{x\neq a} \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)dy \right)^\alpha G^s(x)$, *provided that $G^{(\beta-s)/\alpha}(y)$ makes sense.*

(3) $\mathcal{A}_3 := \sup_{x\neq a} A_3(x;\alpha,\beta,s) := \sup_{x\neq a} \left(\int_{B(a,|x|_a)} g(y)F^{(\alpha-s)/\beta}(y)dy \right)^\beta F^s(x)$, *provided that $F^{(\alpha-s)/\beta}(y)$ makes sense.*

(4) $\mathcal{A}_4 := \sup_{x\neq a} A_4(x;\alpha,\beta,s) := \sup_{x\neq a} \left(\int_{B(a,|x|_a)} f(y)G^{(\beta+s)/\alpha}(y)dy \right)^\alpha G^{-s}(x)$, *provided that $f, g \in L^1(\mathbb{X})$ and that $G^{-s}(x)$ makes sense.*

(5) $\mathcal{A}_5 := \sup_{x\neq a} A_5(x;\alpha,\beta,s) := \sup_{x\neq a} \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} g(y)F^{(\alpha+s)/\beta}(y)dy \right)^\beta F^{-s}(x)$, *provided $f, g \in L^1(\mathbb{X})$ and that that $F^{-s}(x)$ makes sense.*

Moreover, we have the following relations between the above quantities:

- $\mathcal{A}_1 \leq (\max(1, \frac{s}{\beta}))^\alpha \mathcal{A}_2$ *and* $\mathcal{A}_2 \leq (\max(1, \frac{\beta}{s}))^\alpha \mathcal{A}_1$*;*
- $\mathcal{A}_1 \leq (\max(1, \frac{s}{\alpha}))^\beta \mathcal{A}_3$ *and* $\mathcal{A}_3 \leq (\max(1, \frac{\alpha}{s}))^\beta \mathcal{A}_1$*;*
- $(\frac{s}{\beta+s})^\alpha \mathcal{A}_4 \leq \mathcal{A}_1 \leq (1+\frac{s}{\beta})^\alpha \mathcal{A}_4$ *and* $(\frac{s}{\alpha+s})^\beta \mathcal{A}_5 \leq \mathcal{A}_1 \leq (1+\frac{s}{\alpha})^\beta \mathcal{A}_5$.

Proof of Theorem 4.1. $\boxed{\mathcal{A}_1 \approx \mathcal{A}_2}$

We will first consider the case $s \leq \beta$. Then for $|y|_a \geq |x|_a$ we have $G^{(\beta-s)/\alpha}(y) \geq G^{(\beta-s)/\alpha}(x)$. Consequently, we can estimate

$$\begin{aligned} A_2(x;\alpha,\beta,s) &= \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)dy \right)^\alpha G^s(x) \\ &\geq \left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)dy \right)^\alpha G^\beta(x) \\ &= F^\alpha(x)G^\beta(x), \end{aligned}$$

which implies $\mathcal{A}_2 \geq \mathcal{A}_1$. For $s > \beta$, let us first introduce some notation, using the polar decomposition (1.2). First, we denote

$$\begin{aligned} W(x) &:= \int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)dy \\ &= \int_{|x|_a}^{\infty}\int_{\Sigma} f(r,\omega)\lambda(r,\omega)\widetilde{G_1}(r)^{(\beta-s)/\alpha}d\omega dr \\ &= \int_{|x|_a}^{\infty} \widetilde{W}(r)dr \\ &=: \widetilde{W_1}(|x|_a), \end{aligned}$$

where

$$\widetilde{G_1}(r) := \int_0^r \int_{\Sigma} g(s,\sigma)\lambda(s,\sigma)d\sigma ds = \int_0^r \widetilde{G}(s)ds,$$

with $\widetilde{G}(s) := \int_\Sigma g(s,\sigma)\lambda(s,\sigma)d\sigma$, and

$$\widetilde{W}(r) := \int_{\Sigma} f(r,\omega)\lambda(r,\omega)\widetilde{G_1}(r)^{(\beta-s)/\alpha}d\omega.$$

Moreover, we denote

$$\widetilde{F_1}(r) := \int_r^{\infty}\int_{\Sigma}\lambda(s,\sigma)f(s,\sigma)d\sigma ds = \int_r^{\infty}\widetilde{F}(s)ds.$$

Using the function W defined above, we can estimate

$$\begin{aligned} &F^{\alpha}(x)G^{\beta}(x) \\ &= G^{\beta}(x)\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)G^{(s-\beta)/\alpha}(y)W^{(s-\beta)/s}(y)W^{(\beta-s)/s}(y)dy\right)^{\alpha} \\ &= G^{\beta}(x)\left(\int_{|x|_a}^{\infty}\int_{\Sigma}\lambda(r,\omega)f(r,\omega)\widetilde{G_1}^{(\beta-s)/\alpha}(r)\widetilde{G_1}^{(s-\beta)/\alpha}(r)\widetilde{W_1}^{(s-\beta)/s}(r)\widetilde{W_1}^{(\beta-s)/s}(r)d\omega dr\right)^{\alpha} \\ &\leq \left(\sup_{r>|x|_a}\widetilde{G_1}^{(s-\beta)}(r)\widetilde{W_1}^{(s-\beta)\alpha/s}(r)\right)G^{\beta}(x)\left(\int_{|x|_a}^{\infty}\widetilde{W}(r)(\int_r^{\infty}\widetilde{W}(s)ds)^{(\beta-s)/s}dr\right)^{\alpha} \\ &= \left(\sup_{|y|_a>|x|_a} G^s(y)W^{\alpha}(y)\right)^{(s-\beta)/s}\left(\frac{s}{\beta}\right)^{\alpha} G^{\beta}(x)W^{(\beta\alpha)/s}(x) \\ &\leq \left(\frac{s}{\beta}\right)^{\alpha}\left(\sup_{|y|_a>|x|_a} G^s(y)W^{\alpha}(y)\right)^{(1-\beta/s)}\left(\sup_{|x|_a>0} G^s(x)W^{\alpha}(x)\right)^{\beta/s} \\ &\leq \left(\frac{s}{\beta}\right)^{\alpha}\sup_{|x|_a>0} A_2(x;\alpha,\beta,s). \end{aligned}$$

Therefore, we obtain

$$\mathcal{A}_1 \leq \left(\frac{s}{\beta}\right)^{\alpha}\mathcal{A}_2.$$

Hence, we have for every $s > 0$ the inequality

$$\mathcal{A}_1 \leq \left(\max(1,\frac{s}{\beta})\right)^{\alpha}\mathcal{A}_2.$$

Conversely, we have for $s < \beta$,

$$
\begin{aligned}
&G^s(x)W^\alpha(x)\\
&= G^s(x)\Big(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)F^{(\beta-s)/\beta}(y)F^{(s-\beta)/\beta}(y)dy\Big)^\alpha\\
&= G^s(x)\Big(\int_{|x|_a}^\infty\int_\Sigma \lambda(r,\omega)f(r,\omega)\Big(\int_0^r\int_\Sigma \lambda(s,\sigma)g(s,\sigma)dsd\sigma\Big)^{(\beta-s)/\alpha}\\
&\quad\times \widetilde{F_1}^{(\beta-s)/\beta}(r)\widetilde{F_1}^{(s-\beta)/\beta}(r)drd\omega\Big)^\alpha\\
&= G^s(x)\Big(\int_{|x|_a}^\infty\int_\Sigma \lambda(r,\omega)f(r,\omega)\widetilde{G_1}^{(\beta-s)/\alpha}(r)\widetilde{F_1}^{(\beta-s)/\beta}(r)\widetilde{F_1}^{(s-\beta)/\beta}(r)d\omega dr\Big)^\alpha.
\end{aligned}
$$

Consequently, we can estimate

$$
\begin{aligned}
&G^s(x)W^\alpha(x)\\
&\le \Big(\sup_{r>|x|_a}\widetilde{G_1}^{(\beta-s)/\alpha}(r)\widetilde{F_1}^{(\beta-s)/\beta}(r)\Big)^\alpha G^s(x)\Big(\int_{|x|_a}^\infty \widetilde{F}(r)\Big(\int_r^\infty \widetilde{F}(s)ds\Big)^{(s-\beta)/\beta}dr\Big)^\alpha\\
&= \Big(\sup_{r>|x|_a}\widetilde{G_1}^{\beta}(r)\widetilde{F_1}^{\alpha}(r)\Big)^{(\beta-s)/\beta} G^s(x)\Big(\frac{\beta}{s}\Big)^\alpha \widetilde{F_1}^{(\alpha s)/\beta}(r)\\
&\le \Big(\sup_{|y|_a>|x|_a} G^\beta(y)F^\alpha(y)\Big)^{(\beta-s)/\beta}\Big(\frac{\beta}{s}\Big)^\alpha\Big(\sup_{|x|_a>0}G^\beta(x)F^\alpha(x)\Big)^{s/\beta}\\
&\le \Big(\frac{\beta}{s}\Big)^\alpha \sup_{|x|_a>0}A_1(x;\alpha,\beta),
\end{aligned}
$$

which gives $\mathcal{A}_2 \le \left(\frac{\beta}{s}\right)^\alpha\mathcal{A}_1$.

On the other hand, for $s \ge \beta$, when $|y|_a > |x|_a$ we have $G^{(\beta-s)/\alpha}(y) \le G^{(\beta-s)/\alpha}(x)$.

Therefore, we can estimate

$$
\begin{aligned}
A_2(x;\alpha,\beta,s) &= G^s(x)\Big(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)G^{(\beta-s)/\alpha}(y)dy\Big)^\alpha\\
&\le G^s(x)\Big(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\Big)^\alpha G^{(\beta-s)}(x)\\
&= F^\alpha(x)G^\beta(x)
\end{aligned}
$$

i.e. $\mathcal{A}_2 \le \mathcal{A}_1$. Therefore, we have for $s > 0$, the overall estimate

$$
\mathcal{A}_2 \le \Big(\max(1,\frac{\beta}{s})\Big)^\alpha\mathcal{A}_1.
$$

Hence we have also shown that $\mathcal{A}_1 \approx \mathcal{A}_2$.

Next we observe that the proof of $\boxed{\mathcal{A}_1 \approx \mathcal{A}_3}$ follows along the same lines as that of $\mathcal{A}_1 \approx \mathcal{A}_2$, where we just need to interchange the roles of F and G.

$\boxed{\mathcal{A}_1 \approx \mathcal{A}_4}$

Let us denote

$$\begin{aligned}W_0(x) &:= \int_{B(a,|x|_a)} f(y)G^{(\beta+s)/\alpha}(y)dy \\ &= \int_0^{|x|_a}\int_\Sigma \lambda(r,\omega)f(r,\omega)G^{(\beta+s)/\alpha}(r,\omega)d\omega dr \\ &=: \int_0^{|x|_a} \widetilde{W_0}(r)dr,\end{aligned}$$

so that we can write

$$A_4(x;\alpha,\beta,s) = G^{-s}(x)W_0^\alpha(x).$$

We rewrite A_1 as

$$\begin{aligned}A_1(x;\alpha,\beta) &= G^\beta(x)\bigg(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)G^{(\beta+s)/\alpha}(y)G^{-(\beta+s)/\alpha}(y)dy\bigg)^\alpha \\ &= G^\beta(x)\bigg(\int_{|x|_a}^\infty\int_\Sigma \lambda(r,\omega)f(r,\omega)G^{(\beta+s)/\alpha}(r,\omega)G^{-(\beta+s)/\alpha}(r,\omega)d\omega dr\bigg)^\alpha \\ &= G^\beta(x)\bigg(\int_{|x|_a}^\infty \widetilde{G_1}^{-(\beta+s)/\alpha}(r)\frac{d}{dr}\bigg(\int_0^r \widetilde{W_0}(s)ds\bigg)dr\bigg)^\alpha.\end{aligned}$$

We can estimate this by

$$\begin{aligned}&\le G^\beta(x)\bigg(\widetilde{G_1}^{-(\beta+s)/\alpha}(\infty)W_0(\infty) + \frac{(\beta+s)}{\alpha}\int_{|x|_a}^\infty \widetilde{G}(r)(\widetilde{G_1}(r))^{\frac{-(\beta+s)}{\alpha}-1}W_0(r)dr\bigg)^\alpha \\ &\le G^\beta(x)\bigg(\sup_{|y|_a>|x|_a} G^{-s}(y)W_0^\alpha(y)\bigg)\times \\ &\quad\times\bigg(\widetilde{G_1}^{-\beta/\alpha}(\infty) + \frac{(\beta+s)}{\alpha}\int_{|x|_a}^\infty \widetilde{G_1}^{-(\beta/\alpha)-1}(r)\frac{d}{dr}\bigg(\int_0^r \widetilde{G}(s)ds\bigg)dr\bigg)^\alpha \\ &\le G^\beta(x)\sup_{|y|_a>0} A_4(y;\alpha,\beta,s)\bigg(\widetilde{G_1}^{-\beta/\alpha}(\infty) + \frac{(\beta+s)}{\beta}\bigg(\widetilde{G_1}^{-(\beta/\alpha)}(|x|_a) - \widetilde{G_1}^{-\beta/\alpha}(\infty)\bigg)\bigg)^\alpha \\ &= \sup_{|y|_a>0} A_4(y;\alpha,\beta,s)\bigg[\frac{(\beta+s)}{\beta} + \bigg(1-\frac{(\beta+s)}{\beta}\bigg)\bigg(\frac{G(x)}{G(\infty)}\bigg)^{\beta/\alpha}\bigg]^\alpha \\ &\le \bigg(1+\frac{s}{\beta}\bigg)^\alpha \sup_{|y|_a>0} A_4(y;\alpha,\beta,s),\end{aligned}$$

where the expressions like $G(\infty)$ make sense since $g \in L^1(\mathbb{X})$. Therefore, we obtain

$$\mathcal{A}_1 \le (1+s/\beta)^\alpha \mathcal{A}_4.$$

To prove the opposite inequality, we assume that

$$\sup_{|x|_a>0} A_1(x;\alpha,\beta) < \infty.$$

Then we have

$$A_4(x;\alpha,\beta,s) = G^{-s}(x)\bigg(\int_{B(a,|x|_a)} G^{(\beta+s)/\alpha}(y)f(y)dy\bigg)^{\alpha}$$

$$= G^{-s}(x)\bigg(\int_0^{|x|_a} \widetilde{G_1}^{(\beta+s)/\alpha}(r)\frac{d}{dr}\bigg(-\int_r^{\infty}\widetilde{F}(s)ds\bigg)dr\bigg)^{\alpha}$$

$$= G^{-s}(x)\bigg(\widetilde{G_1}^{(\beta+s)/\alpha}(r)\widetilde{F_1}(r)\bigg|_{|x|_a}^{0} + \frac{\beta+s}{\alpha}\int_0^{|x|_a}\widetilde{F_1}(r)\widetilde{G_1}^{(\beta+s)/\alpha-1}(r)\frac{d}{dr}\bigg(\int_0^r \widetilde{G}(s)ds\bigg)dr\bigg)^{\alpha}$$

$$\leq G^{-s}(x)\bigg(\sup_{0<r<|x|_a}\widetilde{G_1}^{\beta}(r)\widetilde{F_1}^{\alpha}(r)\bigg)\bigg(\frac{\beta+s}{\alpha}\int_0^{|x|_a}\widetilde{G_1}^{s/\alpha-1}(r)\frac{d}{dr}(\int_0^r\widetilde{G}(s)ds)dr\bigg)^{\alpha}$$

$$\leq \bigg(\frac{\beta+s}{\alpha}\bigg)^{\alpha}\sup_{|y|_a>0} G^{\beta}(y)F^{\alpha}(y)G^{-s}(x)\bigg(\frac{\alpha}{s}G^{s/\alpha}(x)\bigg)^{\alpha}$$

$$= \bigg(\frac{\beta+s}{s}\bigg)^{\alpha}\sup_{|x|_a>0} A_1(x;\alpha,\beta),$$

where we have used that $f \in L^1(\mathbb{X})$. Hence we have proved that $A_1 \approx A_4$.

The proof of $\boxed{A_1 \approx A_5}$ follows the same lines as that of the case $A_1 \approx A_4$ if we interchange the roles of F and G. □

5. Equivalent conditions for the Hardy inequality

In this section we prove Theorem 2.1 and also give some comments concerning Theorem 2.2. Without loss of generality we can assume that $f \geq 0$. Then we observe that if in Theorem 4.1 we take

$$f(x) = u(x),\ g(x) = v^{1-p'}(x),\ \alpha = \frac{1}{q},\ \beta = \frac{1}{p'},$$

then it follows that we have the equivalence of the quantities

$$\mathcal{D}_1 \approx \mathcal{D}_2 \approx \mathcal{D}_3 \approx \mathcal{D}_4 \approx \mathcal{D}_5.$$

So, we first assume that any one of these equivalent conditions holds true. In particular, $\mathcal{D}_1 < \infty$, and using polar coordinates (1.2), we have for every $a > 0$ that

$$\bigg\{\int_a^{\infty}\int_{\Sigma}\lambda(r,\omega)u(r,\omega)d\omega dr\bigg\}^{\frac{1}{q}}\bigg\{\int_0^a\int_{\Sigma}\lambda(r,\omega)v^{1-p'}(r,\omega)d\omega dr\bigg\}^{\frac{1}{p'}} \leq \mathcal{D}_1. \tag{5.1}$$

We denote

$$h(t) := \bigg(\int_0^t\int_{\Sigma}\lambda(s,\sigma)v^{1-p'}(s,\sigma)dsd\sigma\bigg)^{\frac{1}{pp'}},$$

$$\widetilde{U}_1(t) := \int_{\Sigma}\lambda(t,\omega)u(t,\omega)d\omega,$$

$$F_1(s) := \int_{\Sigma}\lambda(s,\sigma)[f(s,\sigma)v^{\frac{1}{p}}(s,\sigma)h(s)]^p d\sigma,$$

and

$$H_1(t) := \int_0^t \int_\Sigma \lambda(s,\sigma)[v^{\frac{1}{p}}(s,\sigma)h(s)]^{-p'} d\sigma ds.$$

Then using polar coordinates (1.2), Hölder's inequality, and Minkowski's inequality, the left side of (2.1) can be estimated as

$$\begin{aligned}
&\int_{\mathbb{X}} u(x)\bigg(\int_{B(a,|x|_a)} f(y)dy\bigg)^q dx \\
&\leq \int_0^\infty \int_\Sigma \lambda(r,\omega)u(r,\omega)\bigg(\int_0^r \int_\Sigma \lambda(s,\sigma)[f(s,\sigma)v^{\frac{1}{p}}(s,\sigma)h(s)]^p ds d\sigma\bigg)^{\frac{q}{p}} \\
&\quad \times \bigg(\int_0^r \int_\Sigma \lambda(s,\sigma)[v^{\frac{1}{p}}(s,\sigma)h(s)]^{-p'} ds d\sigma\bigg)^{\frac{q}{p'}} d\omega dr \\
&= \int_0^\infty \widetilde{U}_1(r)\bigg(\int_0^r F_1(s)ds\bigg)^{\frac{q}{p}} H_1^{\frac{q}{p'}}(r)dr \\
&\leq \bigg(\int_0^\infty F_1(s)\bigg(\int_s^\infty \widetilde{U}_1(r)H_1^{\frac{q}{p'}}(r)dr\bigg)^{\frac{p}{q}} ds\bigg)^{\frac{q}{p}}. && (5.2)
\end{aligned}$$

Denoting

$$V_1(s) := \int_\Sigma \lambda(s,\sigma)v^{1-p'}(s,\sigma)d\sigma,$$

and using inequality (5.1) we can estimate

$$\begin{aligned}
H_1(t) &= \int_0^t \int_\Sigma \lambda(r,\sigma)[v^{\frac{1}{p}}(r,\sigma)h(r)]^{-p'} d\sigma dr \\
&= \int_0^t \int_\Sigma \lambda(r,\sigma)v^{1-p'}(r,\sigma)\bigg(\int_0^r \int_\Sigma \lambda(\rho,\omega)v^{1-p'}(\rho,\omega)d\rho d\omega\bigg)^{-\frac{1}{p}} dr d\sigma \\
&= \int_0^t V_1(r)\bigg(\int_0^r V_1(\rho)d\rho\bigg)^{-\frac{1}{p}} dr \\
&= p'\bigg(\int_0^t V_1(r)dr\bigg)^{\frac{1}{p'}} \\
&= p'\bigg(\int_0^t \int_\Sigma \lambda(r,\sigma)v^{1-p'}(r,\sigma)drd\sigma\bigg)^{\frac{1}{p'}} \bigg(\int_t^\infty \int_\Sigma \lambda(r,\sigma)u(r,\sigma)drd\sigma\bigg)^{\frac{1}{q}} \\
&\quad \times \bigg(\int_t^\infty \int_\Sigma \lambda(r,\sigma)u(r,\sigma)drd\sigma\bigg)^{-\frac{1}{q}} \\
&\leq p'\mathcal{D}_1\bigg(\int_t^\infty \widetilde{U}_1(s)ds\bigg)^{-\frac{1}{q}}. && (5.3)
\end{aligned}$$

At the same time we can also estimate

$$\begin{aligned}
\int_s^\infty \widetilde{U}_1(t)\Big(\int_t^\infty \widetilde{U}_1(\tau)d\tau\Big)^{-\frac{1}{p'}}dt &= -p\int_s^\infty \frac{d}{dt}\Big(\int_t^\infty \widetilde{U}_1(\tau)d\tau\Big)^{\frac{1}{p}}dt \\
&= p\left(\int_s^\infty \widetilde{U}_1(t)dt\right)^{\frac{1}{p}} \\
&= p\left(\int_s^\infty \int_\Sigma \lambda(t,\omega)u(t,\omega)dtd\omega\right)^{\frac{1}{p}} \\
&= p\left\{\left(\int_s^\infty \int_\Sigma \lambda(t,\omega)u(t,\omega)dtd\omega\right)^{\frac{1}{q}}\right. \\
&\quad \left.\times\left(\int_0^s \int_\Sigma \lambda(t,\omega)v^{1-p'}(t,\omega)dtd\omega\right)^{\frac{1}{p'}}\right\}^{\frac{q}{p}} \\
&\quad \times\left(\int_0^s \int_\Sigma \lambda(t,\omega)v^{1-p'}(t,\omega)dtd\omega\right)^{-\frac{q}{p'p}} \\
&\le p\mathcal{D}_1^{\frac{q}{p}}h^{-q}(s), \qquad (5.4)
\end{aligned}$$

in view of (5.1). Therefore using (5.3) and (5.4) in (5.2), we have

$$\int_{\mathbb{X}} u(x)\Big(\int_{B(a,|x|_a)} f(y)dy\Big)^q dx \le \mathcal{D}_1^q p'^{\frac{q}{p'}} p\Big(\int_{\mathbb{X}} v(x)f(x)^p dx\Big)^{\frac{q}{p}}.$$

Hence, it follows that (2.1) holds with $C \le \mathcal{D}_1(p')^{\frac{1}{p'}}p^{\frac{1}{q}}$ proving one of the relations in (2.2).

Conversely, let us assume that inequality (2.1) holds, and consider the function

$$f(x) = v^{1-p'}(x)\chi_{(0,t)}(|x|_a)$$

for some $t > 0$, and where χ is the cut-off function. With this function, the right hand side of of (2.1) takes the form

$$\left(\int_{\mathbb{X}} v(x)|f(x)|^p dx\right)^{\frac{1}{p}} = \left(\int_{|x|_a\le t} v^{1-p'}(x)dx\right)^{\frac{1}{p}}.$$

At the same time, the left hand side of of (2.1) takes the form

$$\begin{aligned}
\left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)}|f(y)|dy\right)^q u(x)dx\right)^{\frac{1}{q}} &\ge \left(\int_{|x|_a\ge t}\left(\int_{B(a,|x|_a)}|f(y)|dy\right)^q u(x)dx\right)^{\frac{1}{q}} \\
&= \left(\int_{|x|_a\ge t}\left(\int_{|y|_a\le t} v^{1-p'}(y)dy\right)^q u(x)dx\right)^{\frac{1}{q}} \\
&= \left(\int_{|x|_a\ge t} u(x)dx\right)^{\frac{1}{q}}\left(\int_{|y|_a\le t} v^{1-p'}(y)dy\right).
\end{aligned}$$

Altogether the inequality (2.1) takes the form

$$\left(\int_{|x|_a\geq t} u(x)dx\right)^{\frac{1}{q}}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)\leq C\left(\int_{|x|_a\leq t} v^{1-p'}(x)dx\right)^{\frac{1}{p}},$$

which gives $\mathcal{D}_1 \leq C$. Hence, we have the equivalence and the second relation in (2.2).

As for Theorem 2.2, if we take $f(x) = v^{1-p'}$, $g(x) = u(x)$, $\alpha = \frac{1}{p'}$ and $\beta = \frac{1}{q}$ in Theorem (4.1), we find that

$$\mathcal{D}_1^* \approx \mathcal{D}_2^* \approx \mathcal{D}_3^* \approx \mathcal{D}_4^* \approx \mathcal{D}_5^*.$$

Consequently, we can show Theorem 2.2 by the argument similar to that in Section 5 where Theorem 2.1 was proved. We also note that in the case of homogeneous groups, we can actually also derive it from Theorem 2.1 by the involutive change of variables $x \mapsto x^{-1}$.

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